

NOTE ON CLEBSCH'S THEOREM REGARDING THE SECOND
SET OF JACOBIANS DERIVED FROM $n+1$ HOMOGENEOUS
INTEGRAL FUNCTIONS OF n VARIABLES.

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1. The theorem in question, which was originally published in *Crelle's Journ.*, lxi., pp. 355-358, and lxx., pp. 175-181, was enunciated by its author in the following form: *If $u_1, u_2, u_3, \dots$ be $n+1$ homogeneous integral functions of the m th degree in the n variables $x_1, x_2, x_3, \dots$; and $v_1, v_2, \dots, v_{n+1}$ be the set of Jacobians formed from the u 's: and $w_1, w_2, \dots, w_{n+1}$ be the set similarly formed from the v 's; then the w 's differ from the u 's by a common factor only—that is to say*

$$w_1 = M.u_1, \quad w_2 = M.u_2, \quad \dots, \quad w_{n+1} = M.u_{n+1}.$$

Clebsch's mode of establishing it was to show that for any values of the a 's and b 's the determinant

$$\begin{vmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} & \dots & \frac{\partial v_1}{\partial x_n} & a_1 & b_1 \\ \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} & \dots & \frac{\partial v_2}{\partial x_n} & a_2 & b_2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{\partial v_{n+1}}{\partial x_1} & \frac{\partial v_{n+1}}{\partial x_2} & \dots & \frac{\partial v_{n+1}}{\partial x_n} & a_{n+1} & b_{n+1} \\ \cdot & \cdot & \dots & \cdot & \sum a_h u_h & \sum b_h u_h \end{vmatrix}$$

vanishes identically: that consequently

$$\sum b_h u_h \cdot \sum a_k w_k - \sum a_h u_h \cdot \sum b_k w_k = 0;$$

and that finally "by putting the coefficients of the products $a_k b_k$ separately equal to zero" there results

$$u_k w_k - u_k w_k = 0,$$

as desired.

Even as thus epitomized the procedure is seen to be artificial and somewhat arbitrary: and I therefore propose to put on record a quite simple and more direct treatment, giving also as introduction the lemmas on which both demonstrations rest. For the sake of economizing space n will be taken equal to 3.

2. If u_1, u_2, u_3, u_4 be homogeneous functions of the m th degree in x, y, z , and

$$\begin{aligned} v_1 &\equiv J(u_2, u_3, u_4), & v_2 &\equiv -J(u_1, u_3, u_4), \\ v_3 &\equiv J(u_1, u_2, u_4), & v_4 &\equiv -J(u_1, u_2, u_3); \end{aligned}$$

then

$$u_1 v_1 + u_1 v_2 + u_3 v_3 + u_4 v_4 = 0. \quad (\alpha)$$

This is the same as to say that

$$\begin{vmatrix} u_1 & \frac{\partial u_1}{\partial x} & \frac{\partial u_1}{\partial y} & \frac{\partial u_1}{\partial z} \\ u_2 & \frac{\partial u_2}{\partial x} & \frac{\partial u_2}{\partial y} & \frac{\partial u_2}{\partial z} \\ u_3 & \frac{\partial u_3}{\partial x} & \frac{\partial u_3}{\partial y} & \frac{\partial u_3}{\partial z} \\ u_4 & \frac{\partial u_4}{\partial x} & \frac{\partial u_4}{\partial y} & \frac{\partial u_4}{\partial z} \end{vmatrix} = 0,$$

which follows at once from Euler's theorem that

$$x.col_2 + y.col_3 + z.col_4 = m.col_1.$$

3. With the same notation

$$\left. \begin{aligned} \frac{\partial u_1}{\partial x} v_1 + \frac{\partial u_2}{\partial x} v_2 + \frac{\partial u_3}{\partial x} v_3 + \frac{\partial u_4}{\partial x} v_4 &= 0 \\ \frac{\partial u_1}{\partial y} v_1 + \frac{\partial u_2}{\partial y} v_2 + \frac{\partial u_3}{\partial y} v_3 + \frac{\partial u_4}{\partial y} v_4 &= 0 \\ \frac{\partial u_1}{\partial z} v_1 + \frac{\partial u_2}{\partial z} v_2 + \frac{\partial u_3}{\partial z} v_3 + \frac{\partial u_4}{\partial z} v_4 &= 0 \end{aligned} \right\} \quad (\beta)$$

This is the same as to say that the determinant of the preceding paragraph continues to vanish when in place of its first column we substitute a repetition of one of the other columns.

4. With the same notation

$$\left. \begin{aligned} u_1 \frac{\partial v_1}{\partial x} + u_2 \frac{\partial v_2}{\partial x} + u_3 \frac{\partial v_3}{\partial x} + u_4 \frac{\partial v_4}{\partial x} &= 0 \\ u_1 \frac{\partial v_1}{\partial y} + u_2 \frac{\partial v_2}{\partial y} + u_3 \frac{\partial v_3}{\partial y} + u_4 \frac{\partial v_4}{\partial y} &= 0 \\ u_1 \frac{\partial v_1}{\partial z} + u_2 \frac{\partial v_2}{\partial z} + u_3 \frac{\partial v_3}{\partial z} + u_4 \frac{\partial v_4}{\partial z} &= 0 \end{aligned} \right\} \quad (\gamma)$$

Differentiating (a) with respect to one of the independent variables, say x , we have

$$\sum u_h \frac{\partial v_h}{\partial x} + \sum v_h \frac{\partial u_h}{\partial x} = 0,$$

and since, by (β), the second sum here is equal to 0, so also must the first sum.

5. Solving now the equations (γ) for the u 's we obtain at once Clebsch's result: in fact (γ) may be viewed as simply another way of saying that the u 's are proportional to

$$J(v_2, v_3, v_4), \quad -J(v_1, v_3, v_4), \quad J(v_1, v_2, v_4), \quad -J(v_1, v_2, v_3).$$

6. The next requirement is to explain the cause of the efficacy of Clebsch's irrelevant-looking determinant of the $(n+2)$ th order. In doing this we need not confine ourselves to elements that are differential-coefficients: the determinant to be bordered may be any determinant whatever, anything peculiar being introduced along with the border, and indeed with only part of that. Let us take, for example, the general determinant of the fourth order $|a_1 b_2 c_3 d_4|$. The added row of four elements then is

$$e_1, e_2, e_3, e_4$$

where the e 's again are any quantities whatever: but the added column is

$$0, \quad 0, \quad 0, \quad m_1 d_1 + \dots + m_4 d_4, \quad m_1 e_1 + \dots + m_4 e_4;$$

and the new theorem to be established is

$$(\text{c}) \quad \begin{vmatrix} a_1 & a_2 & a_3 & a_4 & . \\ b_1 & b_2 & b_3 & b_4 & . \\ c_1 & c_2 & c_3 & c_4 & . \\ d_1 & d_2 & d_3 & d_4 & \sum m d \\ e_1 & e_2 & e_3 & e_4 & \sum m e \end{vmatrix} = |d_1 e_2| \cdot |D_1 m_2| + |d_1 e_3| \cdot |D_1 m_3| + \dots \\ \dots + |d_3 e_4| \cdot |D_3 m_4|,$$

where D_r is the cofactor of d_r in $|a_1 b_2 c_3 d_4|$.

More proofs than one suggest themselves. That which lends itself most readily to generalization consists in changing the last column into

$$-\Sigma ma, -\Sigma mb, -\Sigma mc, 0, 0,$$

and then expanding in terms of the minors formed from the last two rows. Doing this we see that the cofactor of $|d_1 e_2|$

$$\begin{aligned} &= - \begin{vmatrix} a_3 & a_4 & m_1 a_1 + m_2 a_2 \\ b_3 & b_4 & m_1 b_1 + m_2 b_2 \\ c_3 & c_4 & m_1 c_1 + m_2 c_2 \end{vmatrix} \\ &= -m_1 |a_1 b_3 c_4| - m_2 |a_2 b_3 c_4| = -m_1 D_2 + m_2 D_1 \\ &= |D_1 m_2|; \end{aligned}$$

and similarly with the cofactors of $|d_1 e_3|$, etc.

7. Applying this result to Clebsch's determinant we see that the latter is equal to

$$|a_1 b_2| \cdot |u_1 w_2| + |a_1 b_3| \cdot |u_1 w_3| + \dots;$$

and its equivalent being known to vanish for all values of the a 's and b 's, it follows that the second factor of every term of it must vanish, and that therefore

$$\frac{u_1}{w_1} = \frac{u_2}{w_2} = \frac{u_3}{w_3} = \dots$$

8. The peculiar identity established in §6 is, however, only the first of a series. Thus taking the same basic determinant as before, namely, $|a_1 b_2 c_3 d_4|$, and repeating the process of bordering, we obtain for investigation the determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & . & . \\ b_1 & b_2 & b_3 & b_4 & . & . \\ c_1 & c_2 & c_3 & c_4 & . & . \\ d_1 & d_2 & d_3 & d_4 & \Sigma md & \Sigma nd \\ e_1 & e_2 & e_3 & e_4 & \Sigma me & \Sigma ne \\ f_1 & f_2 & f_3 & f_4 & \Sigma mf & \Sigma nf \end{vmatrix}.$$

Clearly it is equal to

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & \Sigma ma & \Sigma na \\ b_1 & b_2 & b_3 & b_4 & \Sigma mb & \Sigma nb \\ c_1 & c_2 & c_3 & c_4 & \Sigma mc & \Sigma nc \\ d_1 & d_2 & d_3 & d_4 & . & . \\ e_1 & e_2 & e_3 & e_4 & . & . \\ f_1 & f_2 & f_3 & f_4 & . & . \end{vmatrix},$$

and in this the cofactor of $|d_1 e_2 f_3|$ is

$$- \begin{vmatrix} a_4 & m_1 a_1 + m_2 a_2 + m_3 a_3 & n_1 a_1 + n_2 a_2 + n_3 a_3 \\ b_4 & m_1 b_1 + m_2 b_2 + m_3 b_3 & n_1 b_1 + n_2 b_2 + n_3 b_3 \\ c_4 & m_1 c_1 + m_2 c_2 + m_3 c_3 & n_1 c_1 + n_2 c_2 + n_3 c_3 \end{vmatrix},$$

which

$$\begin{aligned} &= -|a_4 b_1 c_2| m_1 n_2 - |a_4 b_1 c_3| m_1 n_3 - |a_4 b_2 c_1| m_2 n_1 \\ &\quad - |a_4 b_2 c_3| m_2 n_3 - |a_4 b_3 c_1| m_3 n_1 - |a_4 b_3 c_2| m_3 n_2, \\ &= D_3 m_1 n_2 - D_2 m_1 n_3 - D_3 m_2 n_1 + D_1 m_2 n_3 + D_2 m_3 n_1 - D_1 m_3 n_2, \\ &= |D_1 m_2 n_3|. \end{aligned}$$

Similarly the cofactors of $|d_1 e_2 f_4|$, $|d_1 e_3 f_4|$, $|d_2 e_3 f_4|$ are found to be $|D_1 m_2 n_4|$, $|D_1 m_3 n_4|$, $|D_2 m_3 n_4|$; so that our double-bordered determinant is expressible as a sum of products of pairs of three-line determinants. (ϵ)

9. As this result may be written

$$\begin{vmatrix} d_1 & d_2 & d_3 & d_4 \\ e_1 & e_2 & e_3 & e_4 \\ f_1 & f_2 & f_3 & f_4 \end{vmatrix} \cdot \begin{vmatrix} D_1 & D_2 & D_3 & D_4 \\ m_1 & m_2 & m_3 & m_4 \\ n_1 & n_2 & n_3 & n_4 \end{vmatrix}, \quad (\epsilon')$$

which again by the multiplication-theorem is equal to

$$\begin{vmatrix} |a_1 b_2 c_3 d_4| & \Sigma d m & \Sigma d n \\ |a_1 b_2 c_3 e_4| & \Sigma e m & \Sigma e n \\ |a_1 b_2 c_3 f_4| & \Sigma f m & \Sigma f n \end{vmatrix}, \quad (\epsilon'')$$

we observe that we might have begun by expanding the original determinant in terms of the minors formed from the first four columns, and thence proceeded through (ϵ'') and (ϵ') to (ϵ).